VIRTUALLY FREE PRO-p GROUPS

WOLFGANG HERFORT AND PAVEL ZALESSKII

ABSTRACT. We prove that in the category of pro-p groups any finitely generated group G with a free open subgroup splits either as an amalgamated free product or as an HNN-extension over a finite p-group. From this result we deduce that such a pro-p group is the pro-p completion of a fundamental group of a finite graph of finite p-groups.

1. Introduction

Let p be a prime number, and let G be a pro-p group containing an open free pro-p subgroup F. If G is torsion free, then, according to the celebrated theorem of Serre established in [17], G itself is free pro-p.

The main objective of the paper is to give a description of virtually free pro-p groups without the assumption of torsion freeness.

Theorem 1.1. *Let G be a finitely generated pro-p group with a free open subgroup F. Then G is the fundamental pro-p group of a finite graph of finite p-groups of order bounded by $|G : F|$.*

This theorem is the pro-p analogue of the description of finitely generated virtually free discrete groups proved by Karrass, Pietrovski and Solitar in [11]. In the characterization of discrete virtually free groups Stallings' theory of ends played a crucial role. In fact the proof of the theorem of Karrass, Pietrovski and Solitar uses the celebrated theorem of Stallings proved in [18] according to which every finitely generated virtually free group splits as an amalgamated free product or HNN-extension over a finite group, respectively. The theory of ends for pro-p groups has been initiated in [12]. However, it is not known whether an analogue of Stallings' Theorem holds in this context. We will prove Theorem 1.1 and such an analogue for finitely generated virtually free pro-p groups using purely combinatorial pro-p group methods combined with results on p-adic representations of finite p-groups.

Date: November 18, 2017.

2010 *Mathematics Subject Classification.* Primary 20E06; Secondary 20E18, 20E08.

Key words and phrases. Virtually free pro-p group, fundamental pro-p group of a finite graph of p-groups.

This research was partially supported by CNPq and CAPES.

Theorem 1.2. *Let G be a finitely generated virtually free pro-p group. Then G is either a non-trivial amalgamated free pro-p product with finite amalgamating subgroup or a non-trivial HNN-extension with finite associated subgroups.*

As a consequence of Theorem 1.1 we obtain that a finitely generated virtually free pro-p group is the pro-p completion of a virtually free discrete group. However, the discrete result is not used (and cannot be used) in the proof.

V.A.Romankov proved in [15] that the automorphism group of a finitely generated free pro-p group $\mathrm{Aut}(\widehat{F_n})$ of rank $n \geq 2$, is infinitely generated. Therefore, one has that, despite the fact that the automorphism group $\mathrm{Aut}(F_n)$ of a free group of rank n embeds naturally in $\mathrm{Aut}(\widehat{F_n})$, it is by no means densely embedded there! Nevertheless, Theorem 1.1 allows us to show that, surprisingly, the number of conjugacy classes of finite p-subgroups in $\mathrm{Aut}(\widehat{F_n})$ is not greater than the corresponding number for $\mathrm{Aut}(F_n)$.

Note that the assumption of finite generation in Theorem 1.1 is essential: there is an example of a split extension $H = F \rtimes D_4$ of a free pro-2 group F of countable rank which cannot be represented as the fundamental pro-2 group of a profinite graph of finite 2-groups (see Example 5.3).

The line of proof is as follows. In Section 3 we use a pro-p HNN-extension to embed a finitely generated virtually free pro-p group G in a split extension $E = F \rtimes K$ of a free pro-p group F and a finite p-group K with a unique conjugacy class of maximal finite subgroups. In Section 4 we prove using an inductive argument the following theorem which connects the structure of any such group $F \rtimes K$ with its action on $M := F/[F, F]$.

Theorem 1.3. *Let E be a semidirect product $E = F \rtimes K$ of a free pro-p group F of finite rank and a finite p-group K. Then the K-module $M = F/[F, F]$ is permutational if and only if F posesses a K-invariant basis.*

This theorem gives an HNN-extension structure on E with finite base group. In particular, E and, therefore, G acts on a pro-p tree with finite vertex stabilizers. Using this, [6, Proposition 14], and a result from [9] on pro-p groups acting on trees we prove in Section 5 Theorems 1.1 and 1.2. Finally, Section 6 deals with automorphisms of a free pro-p group.

Basic material on profinite groups can be found in [19, 13]. Throughout the paper we make the following standard assumptions. Subgroups are closed and homomorphisms are continuous. For elements x, y in a group G we will write $y^x := xyx^{-1}$ and $[x, y] := xyx^{-1}y^{-1}$. For a subset $A \subseteq G$ we denote by $(A)_G$ the normal closure of A in G, i.e., the smallest closed normal subgroup of G containing A. For profinite graphs we will use (standard) notations which can be found in [14]. The *Frattini subgroup* of G will be denoted by $\Phi(G)$, and $\mathrm{Tor}(G)$ will stand for the subset of elements of finite order in G. For a finite p-group G let $\mathrm{socle}(G) := \langle c \in \mathrm{Z}(G) \mid c^p = 1\rangle$ denote the *socle* of G. Modules will be free $\mathbb{Z}_p$-modules of finite rank.

Acknowledgement

The authors would like to thank the referee(s) for helpful remarks that lead to considerable improvement of this paper.

2. Preliminary results

2.1. **Pro-p modules.** Modules will be left modules in the paper.

Theorem 2.1. (*Diederichsen, Heller-Reiner,* [4, (2.6) Theorem]) *Let G be a group of order p and M a $\mathbb{Z}_p[G]$-module, free as a $\mathbb{Z}_p$-module. Then*

$$M = M_1 \oplus M_p \oplus M_{p-1},$$

where M_p is a free G-module, M_1 is a trivial G-module and on M_{p-1} the equality $(1 + c + \cdots + c^{p-1})M_{p-1} = \{0\}$ holds for any generator c of G.

Let G be a p-group. A *permutation lattice* for G (or *G-permutational* module) is a direct sum of G-modules, each of the form $\mathbb{Z}_p[G/H]$ for some subgroup H of G. Note that a G-module M which is a free $\mathbb{Z}_p$-module is a permutation lattice if and only if G permutes the elements of a basis of M. In particular, when $H \leq G$ and M is a G-permutation lattice it is an H-permutation lattice.

If $G = \langle c\rangle$ is of order p, then Theorem 2.1 implies that M is a permutational lattice if and only if M_{p-1} is missing in the decomposition for M if and only if $M/(c-1)M$ is torsion free.

Corollary 2.2. *With the assumptions of Theorem 2.1 suppose that M admits a Heller-Reiner decomposition $M = M_1 \oplus M_p$. Let L be a free G-submodule of M such that M/L is torsion free. There is a free $\mathbb{Z}_p[G]$-submodule M'_p containing L such that $M = M_1 \oplus M'_p$ is a Heller-Reiner decomposition.*

Proof. Consider the canonical epimorphism of G-modules from M onto $\overline{M} := M/pM$. Since M/L is torsion free, one has $pM \cap L = pL$. From this we can deduce that $\overline{L}$ is a free $\mathbb{F}_p[G]$-module, and so it is injective. Therefore there is a G-invariant complement $\overline{N}$ of $\overline{L}$ in $\overline{M}$. Since $\overline{M} = \overline{M}_1 \oplus \overline{M}_p$ by Krull-Schmidt, $\overline{N} = \bigoplus_{i\in I} \overline{N}_i$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules N_i each of them either free or trivial.

Lift each free $\overline{N}_i$ to a cyclic $\mathbb{Z}_p[G]$-submodule N_i, and let $N_p := \sum_{i\in I} N_i$. Put $\overline{M}'_p := \overline{L} \oplus \overline{N}$, and let $M'_p := L + N$. Since the $\mathbb{Z}_p$-rank of M'_p coincides with the $\mathbb{F}_p$-dimension of $\overline{M}'_p$, it must be a free

$\mathbb{Z}_p[G]$-submodule of M and it contains L. Note that $M/M'_p \cong M_1$ by Krull-Schmidt and so one has $M'_p + M_1 = M$.

Let us show that $M_1 \cap M'_p = \{0\}$. There is an idempotent e with $M_1 = eM$ and $(1-e)M = M_p$. Then $eM'_p = M_1 \cap M'_p$ and therefore $M'_p = eM'_p \oplus (1-e)M'_p = (M_1 \cap M'_p) \oplus (1-e)M'_p$. Since M'_p is a free $\mathbb{Z}_p[G]$-module it cannot have the trivial G-module as a non-trivial direct summand. Hence $M_1 \cap M'_p = \{0\}$ as desired and the corollary is proved. □

2.2. **Pro-p modules and pro-p groups.** Let $G := F \rtimes C_p$ be a semidirect product of a finitely generated free pro-p group with a group of order p. We need to relate the Heller-Reiner decomposition of the induced C_p-module $F/[F,F]$, with a specific free product decomposition of G.

Lemma 2.3. *Let G be a split extension of a free pro-p group F of finite rank by a group of order p. Then*

(i) *(*[16]*) G has a free decomposition $G = \left(\coprod_{i\in I}(C_i \times H_i)\right) \amalg H$, with $C_i \cong C_p$ and all H_i and H free pro-p.*

Here I is finite and each C_i is a representative of a conjugacy class of cyclic subgroups of order p in G. The subgroups H_i and H are contained in F and $C_F(C_i) = H_i$.

(ii) *(*[6, Lemma 6]*) Set $M := F/[F,F]$. Fix $i_0 \in I$ and a generator c of C_{i_0}. Then conjugation by c induces an action of C_{i_0} upon M. The $\langle c\rangle$-module M admits a Heller-Reiner decomposition $M = M_1 \oplus M_p \oplus M_{p-1}$.*

Moreover, the $\mathbb{Z}_p$-ranks of the three $\langle c\rangle$-modules satisfy $\mathrm{rank}(M_p) = p\,\mathrm{rank}(H)$, $\mathrm{rank}(M_{p-1}) = (p-1)(|I|-1)$, and $\mathrm{rank}(M_1) = \sum_{i\in I}\mathrm{rank}(H_i)$.

In particular, M is G/F-permutational if and only if $|I| = 1$.

We shall use also the following corollary that can be extracted from [6, Corollary 7].

Corollary 2.4. *If for each $i \in I$ a basis B_i of H_i is given and B is any basis of H, then $\bigcup_{i\in I} B_i[F,F]/[F,F]$ is a basis of M_1 and $B[F,F]/[F,F]$ is a basis of the G/F-module M_p. A basis of M_{p-1} is given by $\{c_{i_0}^{-1}c_i \mid i \in I, i \neq i_0\}[F,F]/[F,F]$.*

Corollary 2.5. *When C_p acts as a group of automorphisms on a finitely generated free pro-p group F and the induced action of C_p on $M := F/[F,F]$ allows an interpretation $M = M_1 \oplus M_p$ as a permutation module, then the image of $C_F(C_p)$ under the commutator quotient map intersects trivially with M_p and has the same $\mathbb{Z}_p$-rank as M_1.*

Proof. Lemma 2.3 implies that $H := (C_p \times C_F(C_p)) \amalg F_0$ for a free pro-p subgroup F_0. The same lemma shows that there is a Heller-Reiner decomposition $M_1' \oplus M_p'$ with $M_1' = C_F(C_p)[F,F]/[F,F]$. Setting in Corollary 2.2 $L := M_p$ implies that $M_1' \cap M_p = \{0\}$, as claimed. The equality of $\mathbb{Z}_p$-ranks follows from Corollary 2.4, noting that $|I| = 1$. □

Lemma 2.6. *Suppose that $G = F \rtimes \langle t \rangle = (\langle t \rangle \times C_F(t)) \amalg F_0$ with $\langle t \rangle \cong C_p$ and Q is a t-invariant free pro-p factor of F with $Q/[Q,Q]$ a free $\langle t \rangle$-module. Let "bar" indicate passing to the quotient modulo $(Q)_F$. Then $C_F(t) \cong C_{\overline{F}}(t) = \overline{C_F(t)}$ and $\overline{G} \cong (C_p \times C_{\overline{F}}(t)) \amalg F_1$ for some free pro-p group F_1.*

Proof. Since Q is a free pro-p factor of F, we find that $Q \cap [F,F] = [Q,Q]$. Therefore by Lemma 2.3 the $\langle t \rangle$-submodule $L := Q[F,F]/[F,F]$, being isomorphic to $Q/[Q,Q]$, is a free $\langle t \rangle$-submodule of $M := F/[F,F]$. Consider a Heller-Reiner decomposition $M = M_1 \oplus M_p$. Since M/L is a free $\mathbb{Z}_p$-module we can, using Corollary 2.2, arrange M_p such that L becomes a direct summand of M_p. Note that $\overline{F}/[\overline{F},\overline{F}] = M/L$. Since $\mathbb{Z}_p[\langle t \rangle]$ is a local ring the Krull-Schmidt theorem applies to the Heller-Reiner decomposition $\overline{F}/[\overline{F},\overline{F}] = N_1 \oplus N_p$ showing that the $\mathbb{Z}_p$-rank of M_1 coincides with the $\mathbb{Z}_p$-rank of N_1. Hence by Corollary 2.5, one has $C_{\overline{F}}(t) \cong C_F(t)$ and so certainly $\overline{C_F(t)} = C_{\overline{F}}(t)$. Moreover, Lemma 2.3 shows that $\overline{G} \cong (C_p \times C_{\overline{F}}(t)) \amalg F_1$ for some free pro-p group F_1. □

2.3. **Helpful facts on pro-p groups.**

Lemma 2.7. *Let $F = (A \amalg B) \amalg C$ be a pro-p group. Then $(A \amalg B) \cap (A)_F = (A)_{A \amalg B}$.*

Proof. Observe that $A \amalg B/((A)_F \cap (A \amalg B)) \cong (A \amalg B)(A)_F/(A)_F \cong B$. As $(A)_{A\amalg B} \leq (A)_F \cap (A \amalg B)$ the second isomorphism theorem reads $(A \amalg B/(A)_{A\amalg B})/(((A \amalg B) \cap (A)_F)/(A)_{A\amalg B}) \cong (A \amalg B)/((A \amalg B) \cap (A)_F) \cong B$. Therefore $B \cong A \amalg B/(A)_{A\amalg B} \cong (A \amalg B)/((A \amalg B) \cap (A)_F)$ so that the canonical epimorphism from $A \amalg B/(A)_{A\amalg B}$ onto $(A \amalg B)/((A \amalg B) \cap (A)_F)$ turns out to be an isomorphism. This shows the Lemma. □

Lemma 2.8. *Let $G = F \rtimes K$ with F free pro-p and K a finite p-group. Suppose that every finite subgroup of G is F-conjugate into K. Then, for any $T \leq K$,*

- *(i) $N_G(T) = C_F(T) \rtimes N_K(T)$;*
- *(ii) Every finite subgroup of $N_G(T)$ is $C_F(T)$-conjugate to a subgroup of $N_K(T)$;*

Proof. (i) observe that $g \in N_G(T)$ can be written as $g = fk$ with $f \in F$ and $k \in K$. Then $T = T^g = T^{fk}$ reads modulo F as $T = T^k$ so that $k \in N_K(T)$ and hence $f \in C_F(T)$ follows.

(ii) Let R be a finite subgroup of $N_G(T)$ and w.l.o.g. we can assume that it contains T (multiplying it by T if necessary). By the hypothesis there exists $f \in F$ with $R^f \leq K$; hence $T^f \leq K$. Therefore $TT^f \leq K$

and, since $F \triangleleft G$, for every element $t \in T$ one has $t^{-1}t^f \in K \cap F$. As $K \cap F = \{1\}$ it follows that $f \in C_F(T)$ as needed. $\square$

Our proof is based on the following results from [6] and [16] frequently used in the paper.

Theorem 2.9. [16, Theorem 1.2] *Let K be a finite p group acting on a free pro-p group F of finite rank. Then $C_F(K)$ is a free pro-p factor of F.*

Theorem 2.10. [6, Proposition 14] *Let G be a semidirect product of a free pro-p group F of finite rank with a p-group K such that every finite subgroup is conjugate to a subgroup of K. Suppose that $C_F(t) = \{1\}$ holds for every torsion element t of G. Then $G = K \amalg F_0$ for a free pro-p factor F_0.*

3. HNN-extensions

We introduce a notion of a *pro-p HNN-extension* as a generalization of the construction described in [14, page 97].

Definition 3.1. Suppose that G is a pro-p group, and for a finite set I there are given monomorphisms $\phi_i : A_i \to G$ for subgroups A_i of G. The *HNN-extension* $\tilde{G} := \mathrm{HNN}(G, A_i, \phi_i, i \in I)$ is defined to be the quotient of $G \amalg F(I)$ modulo the relations $\phi_i(a_i) = ia_i i^{-1}$ for all $i \in I$. We call $\tilde{G}$ an *HNN-extension* and G the *base group*, I the set of *stable letters*, and the subgroups A_i and $B_i := \phi_i(A_i)$ *associated*.

One can see that every HNN-extension in the sense of the present definition can be obtained by successively forming *HNN-extensions*, as defined in [14], each time defining the base group to be the just constructed group and then adding a pair of associated subgroups and a new stable letter.

Remark 3.2. In the presentation of H we may, for every $i \in I$, choose $k_i \in K$ and replace every (A_i, Z_i) by $(B_i, X_i) := (A_i^{k_i}, Z_i^{k_i})$ and Z_i by Z_B. Then $H = \mathrm{HNN}(K, B_i, X_i, I)$.

A pro-p HNN-extension $G = \mathrm{HNN}(H, A, f, t)$ is *proper* if the natural map from H to G is injective. Only proper pro-p HNN-extensions will be used in this paper.

A proper HNN-extension $\tilde{G} := \mathrm{HNN}(G, A_i, \phi_i, I)$ (viewing G as a subgroup of $\tilde{G}$) satisfies a *universal property* as follows. Given a pro-p group G, homomorphisms $f : G \to H$, $f_i : A_i \to H$ and a map $g : I \to H$ such that for all $i \in I$ and all $a_i \in A_i$ we have $f(\phi_i(a_i)) = g(i)f_i(a_i)g(i)^{-1}$, there is a unique homomorphism $\omega : \tilde{G} \to H$ which agrees with f on G, with f_i on A_i for every $i \in I$ and with g on I.

Remark 3.3. Every finite subgroup of $\tilde{G}$ is conjugate to a subgroup of G. This can either be seen by interpreting $\tilde{G}$ as an iterated HNN-extension and then using [14, Theorem 4.2(c)] or by viewing $\tilde{G}$ as the fundamental pro-p group of a graph of groups, the graph being a finite bouquet of loops using [20, Theorem 3.10]

3.1. **HNN-embedding.** Theorem 3.5 below is an HNN-embedding result – a refined pro-p-version of the main theorem in [7]. We first prove it for semidirect products.

Proposition 3.4. *Let $G = F \rtimes K$ be a semidirect product of a free pro-p group F of finite rank and a finite p-group K. Then G can be embedded in a semidirect product $\tilde{G} = E \rtimes K$ such that every finite subgroup of $\tilde{G}$ is conjugate to a subgroup of K and E is free pro-p of finite rank.*

Proof. By [16, Cor. 1.3(a)], there are only finitely many conjugacy classes of finite subgroups that are not conjugate to a subgroup in K. We proceed by induction on this number $f = f(G, K)$. For $f = 0$ there is nothing to prove. For the inductive step it suffices to show that G can be embedded into a semidirect product $\tilde{G}$ of a finitely generated free pro-p group E and (the same) K with less conjugacy classes of finite subgroups that are not conjugate to a subgroup in K. So assume that L is a finite subgroup of G not conjugate to a subgroup of K. Let $\pi : G \longrightarrow K$ be the canonical projection and $\phi = \pi_{|L}$. Put $\tilde{G} := \mathrm{HNN}(G, L, \phi)$ and observe that it is finitely generated.

For proving that G embeds in $\tilde{G}$ we need to employ [1, Theorem 1.3], according to which G embeds in $\tilde{G}$ if, and only if, the following set $\mathcal{N}$ of open normal subgroups intersects trivially: namely $\mathcal{N}$ is the set of all open normal subgroups U of G such that there is a chain of normal subgroups $U = C_0 < \cdots < C_n = G$ with $\phi(L \cap C_i) = \phi(L) \cap C_i$ and ϕ inducing the identity on each $(LC_i \cap C_{i+1})/C_i$ for all $i < n$.

Let us show that every open normal subgroup U of G properly contained in F must belong to $\mathcal{N}$. Consider the chain $C_0 := U$, $C_1 := F$ and $C_2 := G$. The conditions hold in the part below $C_1 = F$ since $L \cap F = \phi(L) \cap F = \{1\}$. It is also trivial that $\phi(L \cap C_2) = \phi(L) \cap C_2$, since $C_2 = G$. So we are left with showing that the homomorphism $\overline{\phi}$ induced by ϕ on LF/F coincides with the identity. For $g \in G$ we denote by $\overline{g}$ its image modulo F. If $\overline{x} \in LF/F$ with $x \in L$, then we have $\overline{\phi}(\overline{x}) = \overline{\phi(x)}$, and since $\phi = \pi_{|L}$, $\overline{\phi}(\overline{x}) = \overline{\pi(x)}$. By the definition of the projection π, if $x = fk$ with $f \in F$ and $k \in K$, then $\pi(x) = k$. Hence $\overline{\phi}(\overline{x}) = \overline{\pi(x)} = \overline{k} = \overline{x}$, as desired.

Note that $\pi : G \longrightarrow K$ extends to $\tilde{G} \longrightarrow K$ by the universal property of an HNN-extension, so $\tilde{G}$ is a semidirect product $E \rtimes K$ of its kernel E with K. By [7, Lemma 10], every open torsion free subgroup of $\tilde{G}$ is free pro-p. So E is free pro-p. As $\tilde{G}$ is finitely generated, E is finitely generated. Let A be any finite subgroup of $\tilde{G}$. Then, by [14, Theorem 4.2.(c)], it is conjugate to a subgroup of the base group. □

Having established the HNN-embedding result for semidirect products we state and prove it for arbitrary finitely generated virtually free pro-p groups.

Theorem 3.5. *Let G be a finitely generated pro-p group possessing an open normal free pro-p subgroup F. Then G can be embedded in a semidirect product $\tilde{G} = E \rtimes G/F$ such that every finite subgroup of $\tilde{G}$ is conjugate to a subgroup of G/F and E is free pro-p. Moreover, $\tilde{G}$ is finitely generated.*

Proof. Put $K := G/F$, and let $\pi\colon G \to K$ denote the canonical projection. Form $G_0 := G \amalg K$. By the universal property of the free pro-p product there is an epimorphism from G_0 to K which agrees with π on G and with the identity on K. As a consequence of the Kurosh subgroup theorem (see [13, Theorem 9.1.9]), its kernel, say F_0, is free pro-p and $G_0 = F_0 \rtimes K$, where K is identified with its image in G_0. One observes that G_0 is finitely generated, since G is. Now the result follows from Proposition 3.4. □

3.2. **Permutation extensions.**

Definition 3.6. Given a finite p-group K and a finite K-set X, there is a natural extension of the action of K to the free pro-p group $\tilde{F} = F(X)$. The semidirect product $\tilde{F} \rtimes K$ will be called the *permutational extension* of $\tilde{F}$ by K. Now K acts on $\tilde{F}$ from the left by conjugation, i.e., $k \cdot f[\tilde{F}, \tilde{F}] := f^k[\tilde{F}, \tilde{F}]$.

Remark 3.7. Choosing representatives $\{A_i \mid i \in I\}$ of the conjugacy classes of all point stabilizers and letting $Z_i \subseteq X$ be a set of representatives of orbits such that $K_z = A_i$ for all $z \in Z_i$, we can rewrite the K-set X in the form $\bigcup_{i\in I} K/A_i \times Z_i$ with K acting on the cosets by left multiplication and on the second factor trivially. Then $\tilde{G} := \tilde{F} \rtimes K$ has a presentation $F(\bigcup_{i\in I} Z_i) \amalg K$ modulo the relations $[a_i, z_i]$ for all $z_i \in Z_i$ and $a_i \in A_i$, with i running through the finite set I. The presentation shows that $\tilde{G}$ is isomorphic to an HNN-extension in the sense of Definition 3.1, with all ϕ_i the identity on the respective group A_i, and with the union $\bigcup_{i\in I} Z_i$ as the set of stable letters. We shall write $\tilde{G} = \mathrm{HNN}(K, A_i, Z_i, i \in I)$ – omitting the ϕ_i from the usual notation of the HNN-extension.

Then $M := \tilde{F}/[\tilde{F}, \tilde{F}]$ is a K-permutation module (see the explanation after Theorem 2.1), i.e. $M = \bigoplus_{i\in I} M_i$ with $M_i := \mathbb{Z}_p[K/A_i \times Z_i]$

Lemma 3.8. *Let $\tilde{F}$ be the normal closure of $F(\bigcup_{i\in I} Z_i)$ in $\tilde{G} = \mathrm{HNN}(K, A_i, Z_i, i \in I)$. For every $i \in I$ choose respectively* *coset representative sets R_i of $K/N_K(A_i)$ and S_i of $N_K(A_i)/A_i$. Then $C_F(A_i) = \coprod_{s\in S_i} F(Z_i)^s$ and*

$$\tilde{F} = \coprod_{i\in I} \coprod_{r\in R_i} C_F(A_i)^r.$$

Proof. As explained in Remark 3.7, one can view $\tilde{G}$ as the quotient of $G := F(\bigcup_{i\in I} Z_i) \amalg K$ modulo the relations $[a_i, z_i]$ for all $z_i \in Z_i$ and $a_i \in A_i$, with i running through the finite set I. By the Kurosh subgroup theorem (see [13, Theorem 9.1.9]) applied to the normal closure N of $F(\bigcup_{i\in I} Z_i)$ in G we have a free pro-p decomposition

$$N = \coprod_{i\in I} \coprod_{r\in R_i} \coprod_{s\in S_i} \coprod_{a\in A_i} F(Z_i)^{asr}.$$

The relations yield $F(Z_i)^a = F(Z_i^a) = F(Z_i)$. Since for $s \in S_i, a \in A_i, z \in Z_i$ one has $[a, z] = 1$ if, and only if, $[a^s, z] = 1$ if and only if $[a, z^{s^{-1}}] = 1$ we have

$$\tilde{F} \rtimes A_i = (A_i \times \coprod_{s\in S_i} F(Z_i)^s) \amalg \coprod_{r\in R_i - \{1\}} \coprod_{s\in S_i} F(Z_i)^{sr} \amalg \coprod_{j\neq i} \coprod_{k\in K} F(Z_j)^k.$$

Set $X := A_i \times \coprod_{s\in S_i} F(Z_i)^s$ and observe that $A_i \leq X \cap X^g$ holds for any $g \in C_{\tilde{F}}(A_i)$. Since by Theorem [13, 9.1.12] $X \cap X^h = 1$ for every $h \notin X$, we deduce that $C_F(A_i) = X$. Thus we proved the first equality that in turn implies the second one.

□

Notation 3.9. For a virtually free pro-p group $G = F \rtimes K$ consider the set of subgroups L of K with $C_F(L) \neq 1$ ordered by inclusion. We say that $L \leq K$ is F-**c** maximal if L is maximal with respect to this ordering.

Lemma 3.10. *Let $G = \mathrm{HNN}(K, A_i, Z_i, I)$ be a permutational extension. Then for every F-**c** maximal subgroup L of K there exist elements $i \in I$ and $k \in K$ such that $L = A_i^k$.*

Proof. As in Definition 3.1, we may consider G as an iterated HNN-extension. By [14, Theorem 4.3(b)], in any such HNN-extension the group $K \cap K^x$ is contained in a conjugate of an associated subgroup for any $x \notin K$. Using this fact repeatedly for $1 \neq x \in C_F(L)$ one has that $L \leq K \cap K^x \leq A_i^g$ for a suitable element $g \in G$. Since $C_F(A_i^g) \neq \{1\}$ and L is F-**c** maximal we can conclude that $L = A_i^g$ for some $g \in G$. On the other hand, $G = F \rtimes K$ and so the canonical epimorphism $\pi : G \to K$ yields $k := \pi(g) \in K$ with $L = A_i^k$. □

The goal of the rest of this subsection is to construct a certain K-permutational free pro-p factor Q of F that will serve as a tool for the induction step in Section 4.

Proposition 3.11. *Let $G = \mathrm{HNN}(K, A_i, Z_i, I)$ be a permutational extension as described in Remark 3.7. Consider a family $(B_j)_{j\in J}$ of pairwise non-conjugate subgroups of K each being an F-$\mathbf{c}$ maximal subgroup of G. Then $Q := \langle C_F(B_j) \mid j \in J\rangle = \coprod_{j\in J}\coprod_{r_j\in R_j} C_F(B_j^{r_j})$ and Q is a free pro-p factor of F, where R_j denotes a set of coset representatives of $K/N_K(B_j)$.*

Proof. Lemma 3.10 and Remark 3.2 allow us to identify the family of subgroups $(B_j)_{j\in J}$ with a subfamily of $(A_i)_{i\in I}$, i.e., to assume that $J \subseteq I$ so that $B_j = A_j$ for all $j \in J$. Then Lemma 3.8 gives the result. □

In the final two lemmata of this section we do not have to assume that G is a permutational extension.

Lemma 3.12. *Let $G = F \rtimes K$ be a semidirect product with F free pro-p of finite rank and K a finite p-group. Suppose that every finite subgroup of G is F-conjugate into K. Then, for any F-$\mathbf{c}$ maximal subgroup L of K the normalizer $N_G(L) = \mathrm{HNN}(N_K(L), L, Z_L)$ is a permutational extension.*

Proof. Consider any $t \in N_K(L) - L$. Then $C_{C_F(L)}(t) = \{1\}$ because otherwise there would be $f \in C_F(L)$, $f \neq 1$, fixed by $\langle L, t\rangle$ contradicting L being F-$\mathbf{c}$ maximal. Hence the induced action of $N_K(L)/L$ on $C_F(L)$ is free. Note that $C_F(L)$ is a free factor of F by Theorem 2.9 and hence is finitely generated. Since all finite subgroups of G are conjugate into K by Lemma 2.8(ii), all finite subgroups of $N_G(L)$ are conjugate into $N_K(L)$. As $L \le K$, all finite subgroups of $N_G(L)/L$ are conjugate into $N_K(L)/L$. Therefore, Theorem 2.10 shows that $C_F(L) \rtimes (N_K(L)/L) = A \amalg F_0$ for some finite p-group A and a finitely generated free pro-p group F_0. Selecting a free pro-p base Y of F_0 we have that $N_G(L)/L \cong \mathrm{HNN}(N_K(L)/L, \{1\}, Y)$. Therefore, for $Z_L := Y$ one has $N_G(L) = \mathrm{HNN}(N_K(L), L, Z_L)$, as claimed. □

Lemma 3.13. *Let $G = F \rtimes K$ with F free pro-p of finite rank and K a finite p-group. Suppose that every finite subgroup of G is F-conjugate into K. Assume further that there is $N_K(L) \le K_0 \lhd K$ such that $F \rtimes K_0$ is a permutational extension. Then*

(i) $Q := \langle C_F(L)^k \mid k \in K\rangle$ is a K-invariant free pro-p factor of F and the subgroup $Q \rtimes K$ of G is a permutational extension.

(ii) rank$(Q) = |X_L||K : N_K(L)|$ where X_L is any $N_K(L)$-invariant free pro-p basis of $C_F(L)$ on which $N_K(L)/L$ acts freely.

Proof. By Lemma 3.12, we know that $N_G(L) = \mathrm{HNN}(N_K(L), L, Z_L)$ is a permutational extension.

If $N_K(L) = K$, then $N_G(L) = Q \rtimes K$ is a permutational extension and (ii) holds.

Suppose now that $N_K(L) < K$. Fix coset representative sets T_L of $N_{K_0}(L)/L$, S of $K_0/N_{K_0}(L)$ and R_0 of K/K_0. Then, as $N_K(L) = N_{K_0}(L)$, we find that $R := R_0 S T_L$ is a set of coset representatives of K/L and, as sets, $R = R_0 \times S \times T_L$. In particular, $\{L^{r_0} \mid r_0 \in R_0\}$ is a maximal set of pairwise K_0-non-conjugate K-conjugates of L. Therefore, applying Proposition 3.11 to the family $\{C_F(L^{sr_0} \mid (r_0, s) \in R_0 \times S\}$ inside the permutational extension $F \rtimes K_0$ one obtains that

$$Q_0 := \coprod_{r_0 \in R_0} \coprod_{s \in S} C_F(L^{sr_0})$$

is a free pro-p factor of F. Finally, by Lemma 3.8, $X_L := \bigcup_{t \in T_L} Z_L^t$ is an $N_K(L)$-invariant free pro-p basis of $C_F(L)$. Then $\bigcup_{r \in R} Z_L^r$ is a K-invariant free pro-p basis of Q_0. Therefore Q_0 is a K-invariant free pro-p factor of F and, as $K = R_0 S T_L L$, we find that $Q = Q_0$ must hold.

For showing (ii) it suffices to observe the equalities

$$\mathrm{rank}(Q) = |R_0||S||T_L||Z_L| = |X_L||K : N_K(L)|.$$

□

4. Lifting permutational representations to $F \rtimes K$

A semidirect product $G = F \rtimes K$, where F is a finitely generated free pro-p and K is a finite p-group, will be called a *PE-group*, if every finite subgroup of G is conjugate into K.

For such a group conjugation of finite subgroups can then be achieved by elements in F. By Remark 3.3, every permutational extension is a PE-group. It is the goal of this section to show that the converse holds as well (cf. Proposition 4.8).

4.1. **Induction engine.** Our next proposition describes properties of a "minimal" counter-example G that is a PE-group but not a permutational extension. These properties will be useful for the proof of Proposition 4.8.

Proposition 4.1. *Let $G = F \rtimes K$ be a PE-group such that any PE-group $F' \rtimes K'$ with either $|K'| < |K|$ or $|K| = |K'|$ and rank$(F') <$ rank(F) is a permutational extension. Suppose further that there exists a K-invariant free pro-p factor Q of F such that $Q \rtimes K$ is a permutational extension, and let $\bar{\ } : F \to F/(Q)_F$ denote the canonical projection. Then the following statements hold:*

(i) $\overline{F} \rtimes K$ is a PE-group;

(ii) For every $T \leq K$ we have $C_{\overline{F}}(T) = \overline{C_F(T)}$.

Proof. Suppose that the proposition is false and G is a counter-example. A series of lemmata will yield a contradiction.

Lemma 4.2. $Z(G) = \{1\}$.

Proof. Suppose that $Z(G) \neq \{1\}$. Then there exists $1 \neq t \in \mathrm{socle}(K)$ with $C_F(t) = F$. We claim that $G/\langle t\rangle$ satisfies (i). Indeed, when R is a finite subgroup of $G/\langle t\rangle$ then its preimage in G, say $\tilde{R}$, is F-conjugate into K. Hence R is F-conjugate into $K/\langle t\rangle$. By the minimality assumption on $|K|$ we can conclude that $\overline{F} \rtimes (K/\langle t\rangle)$ is a PE-group. Therefore (i) holds.

Let T be any subgroup of K. Then, by the minimality assumption on $|K|$, we must have $C_{\overline{F}}(T\langle t\rangle/\langle t\rangle) = \overline{C_F(T\langle t\rangle/\langle t\rangle)}$. Now (ii) follows from the equalities $C_F(T) = C_F(T\langle t\rangle) = C_F(T\langle t\rangle/\langle t\rangle)$.

Hence G is not a counter-example, a contradiction. □

Lemma 4.3. *Let $\{1\} \neq t \in$ socle(K). Then either $Q = C_Q(t)$ or $C_Q(t) = \{1\}$.*

Proof. Set $Q_0 := C_Q(t)$ and note that by Theorem 2.9 it is a free K-invariant factor of Q. We can assume that $Q > Q_0 > \{1\}$, else there is nothing to prove. By assumption $Q \rtimes K$ is a permutational extension and so, by Lemma 2.8(ii), $Q_0 \rtimes K = N_{Q\rtimes K}(t)$ is a PE-group. Since $\mathrm{rank}(Q_0) < \mathrm{rank}(F)$, $Q_0 \rtimes K$ is a permutational extension. If $Q = F$ then $\overline{G} = K$ and so G cannot be a counter-example to the statements of our proposition. Thus $\mathrm{rank}(Q) < \mathrm{rank}(F)$ and therefore $Q/(Q_0)_Q \rtimes K$ is a PE-group. Since $\mathrm{rank}(Q/(Q_0)_Q) < \mathrm{rank}(F)$ the quotient $Q/(Q_0)_Q \rtimes K$ is a permutational extension by our minimality assumption on G. By Theorem 2.9 there is $F_0 \leq Q$ so that $Q = Q_0 \amalg F_0$. Setting in Lemma 2.7 $A := Q_0$, $A \amalg B := Q$ implies that $(Q_0)_Q = (Q_0)_F \cap Q$ and hence $Q/(Q_0)_Q \rtimes K \cong (Q(Q_0)_F/(Q_0)_F) \rtimes K$, showing that the latter group is a permutational extension. Using that $\mathrm{rank}(Q_0) < \mathrm{rank}(F)$ and writing "tilde" for passing to the quotient modulo $(Q_0)_F$ we can deduce that statements (i) and (ii) of the proposition hold for $\tilde{G}$, i.e. $\tilde{G}$ is a PE-group and $\widetilde{C_F(t)}$ is naturally isomorphic to $C_{\tilde{F}}(\tilde{t})$. Since

$$\widetilde{(Q)_F} = (Q)_F(Q_0)_F/(Q_0)_F = (F_0)_F(Q_0)_F/(Q_0)_F = (\tilde{Q})_{\tilde{F}} \tag{1}$$

the second isomorphism theorem implies that $\bar{G}$ is naturally isomorphic to $(\tilde{G})/(\tilde{Q})_{\tilde{F}}$. Then observing that $\mathrm{rank}(\tilde{Q}) = \mathrm{rank}(Q(Q_0)_F/(Q_0)_F) < \mathrm{rank}(Q)$ and the pair $(\tilde{G}, \tilde{Q})$ satisfies all hypotheses of the proposition, we find that $\bar{G}$ satisfies (i) and (ii) of the proposition as well. Therefore, G cannot be a counter-example, a contradiction. □

Lemma 4.4. *K cannot be cyclic of order p.*

Proof. Suppose $K \cong C_p$. Lemma 2.3(i) shows that $G = (C_F(K) \times K) \amalg F_0$ with F_0 free pro-p.

Lemma 4.3 implies that either $Q = C_Q(K)$ or $C_Q(K) = \{1\}$. In the first case $C_F(K) = Q \amalg F_Q$ and so $G/(Q)_G \cong (F_Q \times K) \amalg F_0$. Thus (i) and (ii) hold. The second case has been treated in Lemma 2.6. □

Lemma 4.5. *If there is $t \in socle(K)$ with $C_Q(t) < Q$ then $\overline{C_F(K)} = C_{\overline{F}}(K)$.*

Proof. Using Lemma 4.3 we find that $C_Q(t) = \{1\}$. Lemma 2.6 shows that $C_{\overline{G}}(t) = \overline{C_G(t)}$ is naturally isomorphic to $C_G(t)$. As $t \in K$ we have then $C_{\overline{F}}(K) \cong C_F(K)$ and, as $\overline{C_F(K)} \leq C_{\overline{F}}(K)$, we have established the equality $\overline{C_F(K)} = C_{\overline{F}}(K)$. □

Lemma 4.6. *For any $1 \neq t \in socle(K)$ such that $Q = C_Q(t)$ the centralizer $C_{\overline{G}}(t)$ is naturally isomorphic to $C_G(t)/(Q)_{C_F(t)}$.*

Proof. Applying the Kurosh subgroup theorem (see [5, Proposition 4.1]) to the subgroup $C_F(t)$ of $F = Q \amalg F_Q$ we get that $Q = C_Q(t) = C_F(t) \cap Q$ must be a free pro-p factor of $C_F(t)$. Setting in Lemma 2.7 $A := Q$ and $A \amalg B := C_F(t)$ implies that $C_F(t) \cap (Q)_F = C_F(t) \cap (Q)_{C_F(t)}$ so that $C_{\overline{F}}(t) = C_F(t)(Q)_F/(Q)_F \cong C_F(t)/(C_F(t) \cap (Q)_F) \cong C_F(t)/(Q)_{C_F(t)}$. This equality gives $C_{\overline{G}}(t) \cong C_G(t)/(Q)_{C_F(t)}$. □

Lemma 4.7. *For any counter-example G statement (ii) holds.*

Proof. For $\{1\} \neq T < K$ the minimality assumption on $|K|$ shows that $C_{\overline{F}}(T) = \overline{C_F(T)}$ must hold. So all we need to establish is

$$C_{\overline{F}}(K) = \overline{C_F(K)}. \tag{2}$$

Pick any $1 \neq t \in \mathrm{socle}(K)$ and note that $\langle t \rangle < K$ by Lemma 4.4. By Lemma 4.5 we may assume that $Q = C_Q(t)$.

Then by Lemma 4.6, $C_{\overline{G}}(t)$ is naturally isomorphic to $C_G(t)/(Q)_{C_F(t)}$. Therefore, as $t \in K$,

$$C_{\overline{F}}(K) = C_{C_{\overline{F}}(t)}(K) \cong C_{C_F(t)/(Q)_{C_F(t)}}(K). \tag{3}$$

By Lemma 2.8(ii), every finite subgroup of $C_G(t)$ is $C_F(t)$-conjugate into K. By Lemma 4.2, and Theorem 2.9, $\mathrm{rank}(C_F(t)) < \mathrm{rank}(F)$ and by hypothesis $Q \rtimes K$ is a permutational extension. Hence

$$\begin{array}{rcl} C_{C_F(t)/(Q)_{C_F(t)}}(K) & = & C_{C_F(t)}(K)(Q)_{C_F(t)}/(Q)_{C_F(t)} \\ & = & C_F(K)(Q)_{C_F(t)}/(Q)_{C_F(t)} \\ & \cong & C_F(K)/C_F(K) \cap (Q)_{C_F(t)}. \end{array} \tag{4}$$

Taking $C_F(K) \cap (Q)_{C_F(t)} = C_F(K) \cap (C_F(t) \cap (Q)_F) = C_F(K) \cap (Q)_F$ into account yields

$$\begin{array}{rcl} C_F(K)/C_F(K) \cap (Q)_{C_F(t)} & = & C_F(K)/C_F(K) \cap (Q)_F \\ & \cong & C_F(K)(Q)_F/(Q)_F \\ & = & \overline{C_F(K)} \end{array} \tag{5}$$

Combining (3), (4) and (5) yields the desired equation (2). □

Deriving a final contradiction

In order to produce a final contradiction it suffices to establish (i) by Lemma 4.7.

There must be a finite subgroup R of $\overline{G}$ not $\overline{F}$-conjugate into K. If $|R| < |K|$, then taking $\bar{G}_0 = R\bar{F}$ and G_0 to be its preimage in G we see that $G_0 = F \rtimes (G_0 \cap K)$ is a PE-group and $|G_0 \cap K| < |K|$. Then by the minimality assumption on $|K|$ the group R is $\bar{F}$-conjugate into subgroup of K contradicting the hypothesis on R. Thus we must have $|R| = |K|$. Lemma 4.4 implies that $|K| > p$. Conjugating R with a suitable element in $\overline{F}$ we can achieve that $\{1\} \neq R \cap K$ is a maximal subgroup of K. Therefore, there exists $1 \neq t \in \mathrm{socle}(R) \cap \mathrm{socle}(K)$ with $R \leq C_{\overline{G}}(t)$. Lemma 4.3 implies that we can have only the following two cases:

α): $C_Q(t) = \{1\}$.

β): $C_Q(t) = Q$ is a free pro-p factor of $C_F(t)$.

α) Lemma 2.6 shows that $C_F(t) \cong C_{\overline{F}}(t)$ and so $C_G(t) \cong C_{\overline{G}}(t)$. Therefore there is $R_0 \leq C_G(t)$ with $\overline{R}_0 = R$. Now R is $\overline{F}$-conjugate into K since $R_0 \cong K$ is $C_F(t)$-conjugate into K by the minimality assumption on the rank of F (remember that $\mathrm{rank}(C_F(t)) < \mathrm{rank}(F)$ by Lemma 4.2 and Theorem 2.9).

β) An application of Lemma 4.6 gives the natural isomorphism $C_{\overline{G}}(t) \cong C_G(t)/(Q)_{C_F(t)}$. Lemma 2.8(ii) implies that $C_G(t) = C_F(t) \rtimes K$ is a PE-group. Lemma 4.2, Theorem 2.9 and the minimality assumption on the rank of F show that $C_G(t)/(Q)_{C_F(t)} = C_G(t)/(Q)_{C_F(t)} \rtimes K$ is a PE-group. Therefore, $C_{\overline{G}}(t) = C_{\overline{F}}(t) \rtimes K$ is a PE-group. In particular, R is $C_{\overline{F}}(t)$-conjugate into K, a contradiction. $\square$

4.2. **Permutational extension criterion.**

Proposition 4.8. *Every PE-group $G = F \rtimes K$ is a permutational extension.*

Proof. Suppose that the proposition is false. Then there is a counter-example with K of minimal order. Among all such counter-examples fix one with $\mathrm{rank}(F)$ minimal. If there is no finite F-$\mathbf{c}$ maximal subgroup $\{1\} \neq L \leq K$ then by Theorem 2.10 we find $G = F_0 \amalg K = \mathrm{HNN}(K, 1, Z, 1)$ where Z is a base of F_0, a contradiction. Therefore, we can fix an F-$\mathbf{c}$ maximal subgroup $\{1\} \neq L \leq K$ and set $Q := \langle C_F(L)^k \mid k \in K \rangle$. Observe that Q is K-invariant.

We claim that Q is a free pro-p factor of F and $Q \rtimes K$ is a permutational extension.

Indeed, if $L \triangleleft K$ then $Q = C_F(L)$ and hence by Theorem 2.9 Q is a free pro-p factor of F. Lemma 3.12 shows then that $Q \rtimes K = N_G(L) = \mathrm{HNN}(K, L, Z_L, \{L\})$ is a permutational extension. If $N_K(L) < K$ fix any maximal subgroup K_0 of K containing $N_K(L)$. By the minimality assumption on $|K|$ we can conclude that $F \rtimes K_0$ is a permutational extension and therefore the claim follows from Lemma 3.13(i).

Since $Q \rtimes K$ is a permutational extension Proposition 4.1 implies that $\overline{G} := G/(Q)_F = F/(Q)_F \rtimes K$ is a PE-group. As $\mathrm{rank}(\overline{F}) < \mathrm{rank}(F)$

the minimality assumption on rank(F) implies that

$$\overline{G} = \mathrm{HNN}(K, B_j, Y_j, j \in J) \tag{6}$$

is a permutational extension.

Let S_j be a set of coset representatives of $N_K(B_j)/B_j$. By Lemma 3.8, $C_{\overline{F}}(B_j) = \coprod_{s \in S_j} F(Y_j)^s$. Since $C_{olF}(B_j)$ is projective and, by virtue of Proposition 4.1(ii) $C_{\overline{F}}(B_j) = \overline{C_F(B_j)}$, we can lift Y_j to a subset Z_j of some basis of $C_F(B_j)$.

We devise a "model"-permutational extension $\tilde{G}$ that finally will turn out to be isomorphic to G.

To this end we let $\mathcal{A} = \{(B_j, Y_j) \mid j \in J\} \cup \{L, Z_L)\}$. Form $\tilde{G} := \mathrm{HNN}(K, A, Z_A, (A, Z_A) \in \mathcal{A})$ and consider a bijection ϕ which sends, for all $j \in J$ every $B_j \mapsto B_j$, $Y_j \mapsto Z_j$, $L \mapsto L$ and $Z_L \mapsto Z_L$. Using the universal property of the permutational extension $\tilde{G}$, ϕ extends to an epimorphism from $\tilde{G}$ to G.

Since $\overline{G} = G/(C_F(L)^k \mid k \in K)_F = \mathrm{HNN}(K, B_j, Y_j, j \in J)$ and the latter group is naturally isomorphic to $\tilde{G}/(Z_L)_{\tilde{G}}$, we can conclude that $\ker\phi \leq (Z_L)_{\tilde{G}}$ must hold.

Set $\tilde{F} := \phi^{-1}(F)$ and note that $\tilde{G} = \tilde{F} \rtimes K$. Choose a coset representative set R_L of $K/N_K(L)$ and observe that Proposition 3.11 applied to the family $\{C_{\tilde{F}}(L^r) \mid r \in R_L\}$ yields $\tilde{Q} := \coprod_{r \in R_L} C_{\tilde{F}}(L^r)$. Now choose a coset representative set S_L of $N_K(L)/L$ then Lemma 3.8 shows that $C_{\tilde{F}}(L) = \coprod_{s \in S} F(Z_L^s)$ and so we find

$$\mathrm{rank}(\tilde{Q}) = |Z_L||K : L|. \tag{7}$$

As has been mentioned before $\tilde{F}/(\tilde{Q})_{\tilde{F}} \cong F/(Q)_F$ and so establishing

$$\mathrm{rank}(\tilde{Q}) = \mathrm{rank}(Q) \tag{8}$$

would imply $G \cong \tilde{G}$ giving the final contradiction with $\tilde{G}$ being a permutational extension.

If $N_K(L) < K$, then Lemma 3.13(ii) implies (8). Otherwise $L \triangleleft K$ and thus $Q = C_F(L) \cong C_{\tilde{F}}(L)$ because $N_G(L) = \mathrm{HNN}(K, L, Z_L, \{L\}) \cong N_{\tilde{G}}(L)$ (cf. Lemma 3.12). Hence (8) holds in this case as well. ☐

Theorem 4.9. *Let G be a semidirect product of a finitely generated free pro-p group F and a finite p-group K. The following properties are equivalent:*

- *(i) G is a permutational extension.*
- *(ii) Every finite subgroup of G is conjugate to a subgroup of K.*
- *(iii) $M := F/[F, F]$ is a K-permutation module.*

Proof. (i) $\Rightarrow$ (ii) & (iii). If G is a permutational extension, Remark 3.3 and Remark 3.7 together imply that G is a PE-group and that $F/[F, F]$ is a permutation module.

"(ii) $\Rightarrow$ (i)" has been established in Proposition 4.8.

"(iii) $\Rightarrow$ (ii)". Suppose that (iii) holds but (ii) not. Then there is a counter-example G with $|K|$ minimal. Since M is a K-permutational module it is of the form

$$M := F/[F,F] = \bigoplus_{i \in I} M_i \tag{9}$$

with $M_i = \mathbb{Z}_p[(K/A_i) \times Z_i]$ for subgroups $A_i \leq K$ and some finite sets Z_i. Let R be finite subgroup of G. Note that $|R| = |RF \cap K|$ and M is also $RF \cap K$-permutational. Therefore, if $|R| < |K|$ then, by the minimality assumption on $|K|$, R is conjugate to $FR \cap K$ contradicting to the assumption. Therefore $RF = G$ so that $R \cong K$.

Fix $t \in \mathrm{socle}(R)$. Since M is a $\langle t \rangle$-permutation module, t is conjugate into K, and so we may assume $t \in \mathrm{socle}(K)$. Let $M = M_p \oplus M_1$ be the following Heller-Reiner decomposition for $\langle t \rangle$:

$$M_p := \bigoplus_{i \in I, t \notin A_i} M_i, \quad M_1 := \bigoplus_{i \in I, t \in A_i} M_i.$$

By Lemma 2.3(i), $F = C_F(t) \amalg F_t$ for a suitable free pro-p group F_t. Corollary 2.5 implies that $C_F(t)[F,F]/[F,F]$ intersects M_p trivially and $\mathrm{rank}(C_F(t)) = \mathrm{rank}_{\mathbb{Z}_p} M_1$. The natural epimorphism from $C_F(t)$ to $C_F(t)[F,F]/[F,F]$ factors through the canonical K-module homomorphism from $C_F(t)/[C_F(t), C_F(t)]$ to $C_F(t)[F,F]/[F,F]$. Therefore, by the Krull-Schmidt theorem, $C_F(t)/[C_F(t), C_F(t)]$ and M_1 are isomorphic K-permutation modules. As a consequence, $C_G(t)/\langle t \rangle$ is a permutational extension by the minimality assumption on K and, therefore, so is $C_G(t)$. Since $R \leq C_G(t)$, we may conclude that R is conjugate into K by Remark 3.3. $\square$

5. Proof of the main theorems

In this section we shall use the notation and terminology of the theory of pro-p groups acting on pro-p trees from [14]. This will also be the main source of the references.

Theorem 5.1. *Let G be an infinite finitely generated virtually free pro-p group. Then G acts on a pro-p tree with finite vertex stabilizers.*

Proof. By Theorem 3.5, G embeds into a group $\tilde{G} = E \rtimes G/F$ such that every finite subgroup of $\tilde{G}$ is conjugate to a subgroup of G/F and E is free pro-p.

By Theorem 4.9, $\tilde{G}$ is a permutational extension of E and so, by Remark 3.7, can be written as an HNN-extension $\mathrm{HNN}(G/F, A_i, Z_i, I)$ where the base group G/F and the associated groups in A_i are all finite. Thus $\tilde{G}$ acts on a pro-p tree T such that $T/\tilde{G}$ is a bouquet and all vertex stabilizers are finite (cf. [14, p. 89], for the situation of a single loop). $\square$

Proof of Theorem 1.2.

Proof. By Theorem 5.1, G acts on a pro-p tree with finite vertex stabilizers. Since G is finitely generated, by [9, Theorem A], G splits as either a non-trivial amalgamated free pro-p product with finite amalgamating subgroup or a non-trivial HNN-extension with finite associated subgroups. $\square$

Let $(\mathcal{G}, \Gamma)$ be a finite graph of finite p-groups then the fundamental pro-p group $\Pi_1(\mathcal{G}, \Gamma)$ is just the pro-p completion of the usual fundamental group $\pi_1(\mathcal{G}, \Gamma)$ (cf. [20]). Theorem 5.1 allows to deduce Theorem 1.1 from the [9, Theorem A].

Combining Theorem 1.2, the main result in [9], and the main result of Hillman and Schmidt in [10] we can deduce that a pro-p group of positive deficiency having a finitely generated normal subgroup of infinite index splits into an amalgam or an HNN-extension. A pro-p group has *positive deficiency* if its minimal number of generators is greater than its number of relations, i.e. $dim(H^1(G, \mathbb{F}_p)) - dim(H^2(G, \mathbb{F}_p)) > 0$.

Corollary 5.2. *Let G be a finitely generated pro-p group of positive deficiency and N a nontrivial finitely generated normal subgroup of G of infinite index. Then*

(i) *G splits as an amalgamated free pro-p product or as an HNN-extension over a virtually free pro-p group.*
(ii) *G is the fundamental pro-p group of a finite graph of virtually free pro-p groups.*

Proof. By the main result of [10] either N is procyclic and G/N is virtually free pro-p or N is virtually free pro-p and G/N is virtually procyclic. Thus (i) and (ii) follow from Theorem 1.2 and [9, Theorem A], respectively. $\square$

We conclude this section with an example showing that the finite generation assumption on G in Theorem 1.2 is essential.

Example 5.3. Let A and B be groups of order 2 and $G_0 = \langle A \times B, t \mid tAt^{-1} = B\rangle$ be a pro-2 HNN extension of $A \times B$ with associated subgroups A and B. Note that G_0 admits an automorphism of order 2 that swaps A and B and inverts t. Let $G = G_0 \rtimes C$ be the holomorph. Set $H_0 = \langle \mathrm{Tor}(G_0)\rangle$ and $H = H_0 \rtimes C$. Since G_0 is virtually free pro-2, G and H are virtually free pro-2. The main result in [8] shows that H does not decompose as the fundamental pro-2 group of a profinite graph of finite 2-groups. It follows also from the proof in [8] that H does not split as a amalgamated free pro-2 product or a pro-2 HNN-extension over some finite subgroup.

6. Automorphisms

The following theorem is a consequence of Theorems 3.5 and 4.9:

Theorem 6.1. *Let F_n be a free pro-p group of finite rank n and P a finite p-group of automorphisms of F. Then there is an embedding of holomorphs $F_n \rtimes P \longrightarrow F_m \rtimes P$ such that P permutes the elements of some basis of the free pro-p group F_m.*

For a finite set X the canonical embedding of the discrete free group $\Phi(X)$ into its pro-p-completion $F(X)$ induces an embedding of $\mathrm{Aut}(\Phi(X))$ into $\mathrm{Aut}(F(X))$. This embedding is not dense [15]. The next theorem shows that nevertheless it induces a surjection (but not necessarily injection, cf. [3, Proposition 25]) on the conjugacy classes of finite groups.

Theorem 6.2. *Let $F = F(X)$ be a finitely generated free pro-p group and $\Phi = \Phi(X)$ be a dense abstract free subgroup of F on the same set of generators. Suppose that $A \leq Aut(F)$ is a finite p-group. Then there exists an automorphism $\beta \in Aut(F)$ such that the conjugate A^β is contained in $Aut(\Phi)$.*

Proof. Identifying F with its group of inner automorphisms, we may consider the holomorph $G := F \rtimes A$ as a subgroup of $\mathrm{Aut}(F)$. Since G is a finitely generated virtually free pro-p group, we may use [9, Theorem A] in order to present G as the fundamental pro-p group of a finite graph $(\mathcal{G}, \Gamma)$ of finite p-groups. By [20, Theorem 3.10], every finite subgroup of G is conjugate to a subgroup of a vertex group, so there exists $\beta_0 \in G$ with $A^{\beta_0} \in G(v)$ for some $v \in V(\Gamma)$. Let $\pi_1(\mathcal{G}, \Gamma)$ be the abstract fundamental group of the same graph of groups (cf. e.g., [2]), and set $\Phi_0 := \pi_1(\mathcal{G}, \Gamma) \cap F$. Choose a basis Y of Φ_0. Then Y is a basis of $F(X)$, thus there exists $\alpha \in \mathrm{Aut}(F(X))$ sending X bijectively to Y. For $\beta := \beta_0\alpha^{-1}$, $A^\beta \leq \mathrm{Aut}(\Phi)$. □

Theorem 6.3. *Let F be a free pro-p group of rank n.*

- *(i) The embedding $Aut(\Phi) \leq Aut(F)$ induces a surjection between the conjugacy classes of finite p-subgroups of $Aut(\Phi)$ and $Aut(F)$.*
- *(ii) The $Aut(F)$-conjugacy classes of finite subgroups of $Aut(F)$ of order coprime to p are in one-to-one correspondence with $Aut(F/\Phi(F))$-conjugacy classes of finite subgroups of $Aut(F/\Phi(F)) \cong GL_n(\mathbf{F}_p)$ of order coprime to p.*

Proof. Statement (i) is a consequence of Theorem 6.2.

We begin the proof of (ii) by defining a homomorphism $\lambda : \mathrm{Aut}(F) \to \mathrm{Aut}(F/\Phi(F))$ setting

$$\lambda(\alpha)(f\Phi(F)/\Phi(F)) := \alpha(f)\Phi(F)/\Phi(F).$$

By [13, Lemma 4.5.5], the kernel $K := \ker\lambda$ is a pro-p group. Moreover, λ is an epimorphism, since every automorphism $\alpha \in \mathrm{Aut}(F/\Phi(F))$ can be lifted to an automorphism of F (as a consequence of [13, Lemma 4.5.5]).

Let us first show that every p'-subgroup Q (i.e., coprime to p subgroup) of $\mathrm{Aut}(F/\Phi(F))$ is of the form $Q = \lambda(Q_0)$ for a suitable p'-subgroup Q_0 of $\mathrm{Aut}(F)$. Indeed, $\lambda^{-1}(Q)$ contains the normal p-Sylow subgroup K and, therefore, by the profinite version of the Schur-Zassenhaus theorem [13, 2.3.15], $\lambda^{-1}(Q)$ is a split extension of the pro-p group K by a p'-group Q_0, i.e., $\lambda^{-1}(Q) = K \rtimes Q_0$, and so $Q = \lambda(Q_0)$, as desired.

Next suppose that A and B are p'-subgroups of $\mathrm{Aut}(F)$ so that $\lambda(A)$ and $\lambda(B)$ are conjugate in $\mathrm{Aut}(F/\Phi(F))$. Then there exists $g \in F$ so that $A^g K = BK$. Now K is a closed normal p-Sylow subgroup of BK and $K \cap A^g = K \cap B = \{1\}$ shows that A^g and B are complements of K in BK. Therefore, again by [13, Theorem 2.3.15], they are conjugates in BK. Hence A and B are conjugate in G. □

References

[1] Z.M. Chatzidakis, Some remarks on profinite HNN extensions, Isr. J. Math. **85**, No.1-3, (1994) 11-18.

[2] W. Dicks, Groups, Trees and Projective Modules, Springer 1980.

[3] F. Grunewald and P.A. Zalesskii, Genus for groups J. Algebra **326** (2011) 130–168.

[4] A. Heller and I. Reiner, Representations of cyclic groups in rings of integers. I, Ann. of Math. (2) **76** 1962 73–92.

[5] W. Herfort and L. Ribes, Subgroups of free pro-p-products, Math. Proc. Cambridge Philos. Soc. **101** (1987), no. 2, 197–206.

[6] W. Herfort and P.A. Zalesskii, Virtually free pro-p groups whose torsion elements have finite centralizers, Bull. London Math. Soc. 2008 **40** (2008) 929–936.

[7] W. Herfort and P.A. Zalesskii, Profinite HNN-constructions, J. Group Theory **10**(6) (2007) 799–809.

[8] W. Herfort and P.A. Zalesskii, A virtually free pro-p group need not be the fundamental group of a profinite graph of finite groups, Arch. d. Math. **94** (2010), 35–41

[9] W. Herfort, P.A. Zalesskii and T. Zapata, Splitting theorems for pro-p groups acting on pro-p trees and 2-generated subgroups of free pro-p products with procyclic amalgamations (arXiv:1103.2955; submitted)

[10] J. Hillman and A. Schmidt, *Pro-p groups of positive deficiency*, Bulletin of LMS **40** (2008), 1065–1069.

[11] A. Karrass, A. Pietrovski and D. Solitar, Finite and infinite cyclic extensions of free groups, J.Australian Math.Soc. **16** (1973) 458–466.

[12] A.A. Korenev, Pro-p groups with a finite number of ends, Mat. Zametki **76** (2004), no. 4, 531–538; translation in Math. Notes **76** (2004), no. 3-4, 490–496.

[13] L. Ribes and P.A. Zalesskii, Profinite Groups, Springer 2000.

[14] L. Ribes and P.A. Zalesskii, Pro-p Trees and Applications, (2000), Chapter, Ser. Progress in Mathematics, Birkhäuser Boston (2000), Ed. A. Shalev, D. Segal.

[15] L. Roman'kov, Infinite generation of automorphism groups of free pro-p groups, Siberian Mathematical Journal **34**, (1993) 727-732.

[16] C. Scheiderer, The structure of some virtually free pro-p groups, Proc. Amer. Math. Soc. **127** (1999) 695-700.

[17] J.-P. Serre, Sur la dimension cohomologique des groupes profinis, Topology **3**, (1965) 413-420.
[18] J.R. Stallings, Applications of categorical algebra. Proc. Sympos. Pure Math., Vol. XVII. American Mathematical Society, **124** (1970) 124-129.
[19] J.S. Wilson, Profinite Groups, London Math.Soc. Monographs (Clarendon Press, Oxford, 1998)
[20] P.A. Zalesskii and O.V. Mel'nikov, Subgroups of profinite groups acting on trees, Math. USSR Sbornik **63** (1989) 405-424.

University of Technology at Vienna, Austria, w.herfort@tuwien.ac.at
University of Brasilia, Brasilia-DF, Brazil, pz@mat.unb.br

University of Technology at Vienna, Austria

Department of Mathematics, University of Brasilia, Brazil
E-mail address: `w.herfort@tuwien.ac.at`
E-mail address: `pz@mat.unb.br`